Journal of Approximation Theory **105**, 279–291 (2000)
doi:10.1006/jath.2000.3470, available online at http://www.idealibrary.com on IDEAL®

Convergence of Gaussian Quadrature Formulas[1]

Ying Guang Shi

Department of Mathematics, Hunan Normal University, Changsha, Hunan, People's Republic of China
E-mail: syg@lsec.cc.ac.cn

Communicated by Borislav Bojanov

Received April 30, 1999; accepted January 31, 2000

Convergence of a general Gaussian quadrature formula is shown and its rate of convergence is also given.

Key Words: Gaussian quadrature formulas; convergence; rate of convergence.

1. INTRODUCTION AND MAIN RESULTS

This paper deals with convergence of Gaussian quadrature formulas. Let α be a nondecreasing function on $[-1, 1]$ with infinitely many points of increase such that all moments of $d\alpha$ are finite. We call $d\alpha$ a measure. As usual, for $N \in \mathbb{N}$ let $\mathbf{P}_N$ denote the set of polynomials of degree at most N. In what follows we denote by $c, c_1, \ldots$ positive constants independent of variables and indices, unless otherwise indicated; their value may be different at different occurrences, even in subsequent formulas.

Let $n \in \mathbb{N}$. Assume that $m_{0n} \geqslant 0$, $m_{n+1,n} \geqslant 0$, $m_{kn} > 0$, $1 \leqslant k \leqslant n$, are integers, which satisfy $M = \max_{0 \leqslant k \leqslant n+1, n \in \mathbb{N}} m_{kn} < \infty$. Put $N_n = \sum_{k=0}^{n+1} m_{kn} - 1$ and

$$n_0 = \begin{cases} 1, & m_{0n} = 0, \\ 0, & m_{0n} > 0, \end{cases} \qquad n_1 = \begin{cases} n, & m_{n+1,n} = 0, \\ n+1, & m_{n+1,n} > 0. \end{cases}$$

Given a system of nodes

$$1 = x_{0n} > x_{1n} > x_{2n} > \cdots > x_{nn} > x_{n+1,n} = -1 \tag{1.1}$$

[1] Project 19671082 supported by National Natural Science Foundation of China.

0021-9045/00 $35.00

denote by A_{jk}, $0 \leqslant j \leqslant m_k - 1$, $n_0 \leqslant k \leqslant n_1$, the fundamental polynomials for the Hermite interpolation, i.e., $A_{jk} \in \mathbf{P}_{N_n}$ satisfy

$$A_{jk}^{(p)}(x_q) = \delta_{jp}\, \delta_{kq}, \qquad p = 0, 1, ..., m_q - 1,$$

$$j = 0, 1, ..., m_k - 1, \qquad q, k = n_0, n_0 + 1, ..., n_1.$$

The Hermite interpolation of $f \in C^{M-1}[-1, 1]$ is given by

$$H_{N_n}(f, x) = \sum_{k=n_0}^{n_1} \sum_{j=0}^{m_k - 1} f^{(j)}(x_k)\, A_{jk}(x),$$

from which we can obtain the generalized quadrature formula

$$\int_{-1}^{1} f(x)\, \sigma_n(x)\, d\alpha(x) = \sum_{k=n_0}^{n_1} \sum_{j=0}^{m_k - 1} \lambda_{jk}\, f^{(j)}(x_k), \tag{1.2}$$

exact for all $f \in \mathbf{P}_{N_n}$, where

$$\sigma_n(x) = \operatorname{sgn} \prod_{k=1}^{n} (x - x_k)^{m_k}$$

and

$$\lambda_{jk} = \int_{-1}^{1} A_{jk}(x)\, \sigma_n(x)\, d\alpha(x), \qquad j = 0, 1, ..., m_k - 1, \quad k = n_0, n_0 + 1, ..., n_1. \tag{1.3}$$

Particularly interesting is the case when the x_{kn} happen to be the solution $x_{kn}(d\alpha)$ of the extremal problem,

$$\int_{-1}^{1} \left| \prod_{k=n_0}^{n_1} (x - x_k(d\alpha))^{m_k} \right| d\alpha(x)$$
$$= \min_{1 = t_0 \geqslant t_1 \geqslant \cdots \geqslant t_{n+1} = -1} \int_{-1}^{1} \left| \prod_{k=n_0}^{n_1} (x - t_k)^{m_k} \right| d\alpha(x). \tag{1.4}$$

According to [6, Theorem 3] the solution of the extremal problem (1.4) admits the generalized Gaussian quadrature formula

$$\int_{-1}^{1} f(x)\, \sigma_n(x)\, d\alpha(x) = \sum_{k=n_0}^{n_1} \sum_{j=0}^{m_k^*} \lambda_{jk}(d\alpha)\, f^{(j)}(x_k(d\alpha)), \tag{1.5}$$

which is exact for all $f \in \mathbf{P}_{N_n}$, where

$$m_k^* = \begin{cases} m_k - 2, & 1 \leqslant k \leqslant n, \\ m_k - 1, & \text{otherwise} \end{cases}$$

and $\lambda_{jk}(d\alpha) := \lambda_{jkn}(d\alpha)$ are called the Cotes numbers.

Let the sequence of integers $\{r_{kn}\}$ satisfy

$$m_{kn}^* \geqslant r_{kn} \geqslant 0, \qquad n_0 \leqslant k \leqslant n_1, \quad n \in \mathbb{N}.$$

Put

$$r = \max_{n_0 \leqslant k \leqslant n_1,\, n \in \mathbb{N}} r_{kn} \tag{1.6}$$

and

$$Q_n(d\alpha; f) = \sum_{k=n_0}^{n_1} \sum_{j=0}^{r_k} \lambda_{jk}(d\alpha)\, f^{(j)}(x_k(d\alpha)), \qquad f \in C^r[-1, 1]. \tag{1.7}$$

The main aim of this paper is to give conditions of convergence and rate of convergence for the truncated Gaussian quadrature formula $Q_n(d\alpha; f)$ under the assumption that all m_{kn}, $1 \leqslant k \leqslant n$, $n \in \mathbb{N}$, are even. Of course, in this case $\sigma_n = 1$, a.e.

THEOREM 1. *Assume that all m_{kn}, $1 \leqslant k \leqslant n$, $n \in \mathbb{N}$, are even. Let $d\alpha$ be a measure on $[-1, 1]$ such that $\alpha \in C[-1, 1]$. Then*

$$\lim_{n \to \infty} Q_n(d\alpha; f) = \int_{-1}^{1} f(x)\, d\alpha(x), \qquad f \in C^r[-1, 1]. \tag{1.8}$$

In particular,

$$\lim_{n \to \infty} \sum_{k=n_0}^{n_1} \lambda_{0kn}(d\alpha)\, f(x_{kn}(d\alpha)) = \int_{-1}^{1} f(x)\, d\alpha(x), \qquad f \in C[-1, 1] \tag{1.9}$$

and

$$\lim_{n \to \infty} \sum_{k=n_0}^{n_1} \sum_{j=1}^{r_{kn}} \lambda_{jkn}(d\alpha)\, f^{(j)}(x_{kn}(d\alpha)) = 0, \qquad f \in C^r[-1, 1]. \tag{1.10}$$

This result is very general; the special case when $m_0 = m_{n+1} = 0$, $m_1 = \cdots = m_n = 2$ can be found in [7, Theorem 15.2.3, p. 342].

As usual, denote by $\omega(f; \cdot)$ the modulus of continuity of f and

$$\Delta_n(x) = \frac{(1-x^2)^{1/2}}{n} + \frac{1}{n^2}.$$

Let $d_{n_0,n} = |x_{n_0,n} - x_{n_0+1,n}|$, $d_{n_1,n} = |x_{n_1,n} - x_{n_1-1,n}|$, and $d_{kn} = \max\{|x_{kn} - x_{k-1,n}|, |x_{kn} - x_{k+1,n}|\}$, $n_0 + 1 \leqslant k \leqslant n_1 - 1$.

The following general results concern the rate of convergence for $Q_n(d\alpha; f)$.

THEOREM 2. *Assume that all* m_{kn}, $1 \leqslant k \leqslant n$, $n \in \mathbb{N}$, *are even. Let* $d\alpha$ *be a measure on* $[-1, 1]$ *and* $f \in C^r[-1, 1]$. *If*

$$r_{kn} = \min\{r, m_{kn}^*\}, \qquad n_0 \leqslant k \leqslant n_1, \qquad n \in \mathbb{N}, \tag{1.11}$$

then

$$\left| Q_n(d\alpha; f) - \int_{-1}^{1} f(x)\, d\alpha(x) \right| \\ \leqslant cn^{-r}\omega(f^{(r)}; 1/n) \sum_{k=n_0}^{n_1} \lambda_{0kn}(d\alpha) \sum_{j=0}^{m_{kn}^*} d_{kn}^j\, \Delta_n(x_{kn}(d\alpha))^{-j}. \tag{1.12}$$

THEOREM 3. *Assume that all* m_{kn}, $1 \leqslant k \leqslant n$, $n \in \mathbb{N}$, *are even. Let* $x_{kn}(d\alpha) = \cos\theta_{kn}$, $k = 0, 1, ..., n+1$, *and* $f \in C^r[-1, 1]$. *If* (1.11) *holds and*

$$\theta_{k+1,n} - \theta_{kn} \leqslant \frac{c}{n}, \qquad k = 0, 1, ..., n, \tag{1.13}$$

then

$$\left| Q_n(d\alpha; f) - \int_{-1}^{1} f(x)\, d\alpha(x) \right| \leqslant cn^{-r}\omega(f^{(r)}; 1/n). \tag{1.14}$$

In the next section some auxiliary lemmas are established and in the last section the proofs of the theorems are given.

2. AUXILIARY LEMMAS

First we state some known results needed later.

LEMMA A [3]. *For every* $f \in C^p[-1, 1]$ $(p \geqslant 0)$ *there exists a polynomial* $P_n \in \mathbf{P}_n$ *such that for all* $x \in [-1, 1]$

$$|f^{(j)}(x) - P_n^{(j)}(x)| \leqslant c\, \Delta_n(x)^{p-j}\, \omega(f^{(p)}; \Delta_n(x)), \qquad 0 \leqslant j \leqslant p, \tag{2.1}$$

$$|P_n^{(j)}(x)| \leqslant c\, \Delta_n(x)^{p-j}\, \omega(f^{(p)}; \Delta_n(x)), \qquad j > p. \tag{2.2}$$

LEMMA B [4, Theorem 1]. *Assume that all m_{kn}, $1 \leqslant k \leqslant n$, $n \in \mathbb{N}$, are even. We have the generalized Markov–Stieltjes inequality*

$$\sum_{k=i+1}^{n+1} \lambda_{0kn}(d\alpha) \leqslant \int_{-1}^{x_{in}(d\alpha)} d\alpha(x) \leqslant \sum_{k=i}^{n+1} \lambda_{0kn}(d\alpha), \qquad 1 \leqslant i \leqslant n. \tag{2.3}$$

To give an explicit formula for A_{jk}, $0 \leqslant j \leqslant m_k - 1$, $n_0 \leqslant k \leqslant n_1$, set

$$L_k(x) = \prod_{q=n_0,\, q \neq k}^{n_1} \left(\frac{x - x_q}{x_k - x_q} \right)^{m_q}, \tag{2.4}$$

$$b_{\nu k} = \frac{1}{\nu!} [L_k(x)^{-1}]^{(\nu)}_{x=x_k}, \qquad \nu = 0, 1, ..., m_k - 1, \tag{2.5}$$

$$B_{jk}(x) = \sum_{\nu=0}^{m_k - j - 1} b_{\nu k}(x - x_k)^{\nu}. \tag{2.6}$$

Then we have [5, (1.4)]

$$A_{jk}(x) = \frac{1}{j!} (x - x_k)^j\, B_{jk}(x)\, L_k(x),$$

$$j = 0, 1, ..., m_k - 1, \quad k = n_0, n_0 + 1, ..., n_1. \tag{2.7}$$

The following result improves [5, Theorem 1] given by the author and plays a crucial role in this paper.

LEMMA 1. *Assume that $m_0 = m_{n+1} = 0$ and $1 = x_0 \geqslant x_1 > \cdots > x_n \geqslant x_{n+1} = -1$. Let B_{jk} be defined by (2.6). If $m_k - j$ is odd and $0 \leqslant j < i \leqslant m_k - 1$ then*

$$B_{jk}(x) \geqslant c \left| \frac{x - x_k}{c_k} \right|^{i-j} |B_{ik}(x)|, \qquad x \in \mathbb{R}, \quad 1 \leqslant k \leqslant n; \tag{2.8}$$

if j is even and $0 \leqslant i < j \leqslant m_k - 1$, then

$$b_{jk} \geqslant c d_k^{i-j}\, |b_{ik}|, \qquad 1 \leqslant k \leqslant n. \tag{2.9}$$

Moreover,

$$B_{j1}(x) \geqslant c \left|\frac{x-x_1}{d_1}\right|^{i-j} B_{i1}(x) \geqslant 0, \qquad x \leqslant x_1, \qquad 0 \leqslant j < i \leqslant m_1 - 1, \tag{2.10}$$

$$B_{jn}(x) \geqslant c \left|\frac{x-x_n}{d_n}\right|^{i-j} B_{in}(x) \geqslant 0, \qquad x \geqslant x_n, \qquad 0 \leqslant j < i \leqslant m_n - 1, \tag{2.11}$$

and

$$\begin{cases} (-1)^j b_{j1} \geqslant (-1)^i c d_1^{i-j} b_{i1} > 0, & 0 \leqslant i < j \leqslant m_1 - 1; \\ b_{jn} \geqslant c d_n^{i-j} b_{in} > 0, & 0 \leqslant i < j \leqslant m_n - 1. \end{cases} \tag{2.12}$$

Proof. Inequalities (2.8) and (2.9) are already given in [5, Theorem 1]. Meanwhile (2.8) implies (2.10) if $m_1 - j$ is odd and (2.11) if $m_n - j$ is odd. So it is enough to show (2.10) for $m_1 - j$ being even and (2.11) for $m_n - j$ being even. To this end, following the idea of [5], put

$$L_k^*(x) = L_k(x) \left(\frac{x - x_p}{x_k - x_p}\right)^{-1}, \qquad p \neq k.$$

Thus

$$b_{\nu k}^* = \frac{1}{\nu!} [L_k^*(x)^{-1}]_{x=x_k}^{(\nu)} = b_{\nu k} + \frac{1}{x_k - x_p} b_{\nu-1,k}, \qquad \nu \geqslant 1,$$

from which by (2.6) it follows that

$$B_{jk}^*(x) = \sum_{\nu=0}^{m_k-j-1} b_{\nu k}^*(x-x_k)^\nu = B_{jk}(x) + \frac{x-x_k}{x_k-x_p} B_{j+1,k}(x). \tag{2.13}$$

But by (2.8) and the inequalities given in [5, (2.9)]

$$(-1)^\nu b_{\nu 1} > 0, \qquad \nu = 0, 1, ..., m_1 - 1; \quad b_{\nu n} > 0, \quad \nu = 0, 1, ..., m_n - 1, \tag{2.14}$$

we have

$$B_{jk}^*(x) = B_{j+1,k}^*(x) + b_{m_k-j-1}^*(x-x_k)^{m_k-j-1} \geqslant 0,$$

if $k=1$ and $x \leqslant x_1$ or if $k=n$ and $x \geqslant x_n$. This, by means of (2.13) with $k=1$ and $p=2$ or with $k=n$ and $p=n-1$, gives

$$B_{j1}(x) \geqslant c \left|\frac{x-x_1}{d_1}\right| B_{j+1,1}(x) \geqslant 0, \qquad x \leqslant x_1$$

and

$$B_{jn}(x) \geqslant c \left| \frac{x - x_n}{d_n} \right| B_{j+1,\,n}(x) \geqslant 0, \qquad x \geqslant x_n,$$

respectively. Applying these inequalities and (2.8) alternatively several times we can get (2.10) and (2.11).

Comparing the leading coefficients of both the sides of (2.10) and (2.11) as well as using (2.14) yields (2.12). ▮

Using (2.8), (2.10), and (2.11) we can get the important inequalities for $\lambda_{jk}(d\alpha)$.

LEMMA 2. *If* $m_{kn} - j$ *is even and* $0 \leqslant j < i \leqslant m^*_{kn}$, *then*

$$|\lambda_{ikn}(d\alpha)| \leqslant c\sigma_n(x_{kn}(d\alpha) + 0)\, d_{kn}^{i-j} \lambda_{jkn}(d\alpha), \qquad 1 \leqslant k \leqslant n. \tag{2.15}$$

If $m_{0n} > 0$ *then*

$$0 < (-1)^i \lambda_{i0n}(d\alpha) \leqslant (-1)^j c d_{0n}^{i-j} \lambda_{j0n}(d\alpha), \qquad 0 \leqslant j < i \leqslant m^*_{0n}; \tag{2.16}$$

if $m_{n+1,\,n} > 0$ *then*

$$0 < \sigma_n(-1+0)\, \lambda_{i,\,n+1,\,n}(d\alpha) \leqslant c\sigma_n(-1+0)\, d_{n+1,\,n}^{i-j} \lambda_{j,\,n+1,\,n}(d\alpha),$$
$$0 \leqslant j < i \leqslant m^*_{n+1,\,n}. \tag{2.17}$$

Proof. By (1.3), (2.6), and (2.7) for $0 \leqslant p \leqslant m^*_k$

$$\begin{aligned} \lambda_{pk} &= \frac{1}{p!} \int_{-1}^{1} (x - x_k)^p\, B_{pk}(x)\, L_k(x)\, \sigma_n(x)\, d\alpha(x) \\ &= \frac{1}{p!} \int_{-1}^{1} (x - x_k)^p\, B_{p+1,\,k}(x)\, L_k(x)\, \sigma_n(x)\, d\alpha(x) \\ &\quad + \frac{1}{p!}\, b_{m_k - p - 1,\,k} \int_{-1}^{1} (x - x_k)^{m_k - 1}\, L_k(x)\, \sigma_n(x)\, d\alpha(x). \end{aligned}$$

By (1.5) the last term in the above relation is zero. Thus

$$\lambda_{pk} = \frac{1}{p!} \int_{-1}^{1} (x - x_k)^p\, B_{p+1,\,k}(x)\, L_k(x)\, \sigma_n(x)\, d\alpha(x). \tag{2.18}$$

It is easy to see that

$$\sigma_n(x_k+0) = \operatorname{sgn} \prod_{q=1,\, q \neq k}^{n} (x_k - x_q)^{m_q}.$$

Since $m_k - j$ is even, applying (2.18) and (2.8) we have

$$\begin{aligned} \sigma_n(x_k+0)\, \lambda_{jk} &= \frac{\sigma_n(x_k+0)}{j!} \int_{-1}^{1} (x-x_k)^j\, B_{j+1,k}(x)\, L_k(x)\, \sigma_n(x)\, d\alpha(x) \\ &= \frac{1}{j!} \int_{-1}^{1} B_{j+1,k}(x)\, |(x-x_k)^j\, L_k(x)|\, d\alpha(x) \\ &\geqslant c d_k^{j-i} \left| \frac{1}{i!} \int_{-1}^{1} (x-x_k)^i\, B_{i+1,k}(x)\, L_k(x)\, \sigma_n(x)\, d\alpha(x) \right| \\ &= c d_k^{j-i}\, |\lambda_{ik}|. \end{aligned}$$

This proves (2.15).

To prove (2.16) and (2.17) we need [1, Lemma 2], which says that if $m_0 > 0$ then

$$(-1)^p\, \lambda_{p0}(d\alpha) > 0, \qquad 0 \leqslant p \leqslant m_0^*$$

and if $m_{n+1} > 0$ then

$$\sigma_n(-1+0)\, \lambda_{p,n+1}(d\alpha) > 0, \qquad 0 \leqslant p \leqslant m_{n+1}^*.$$

Using these relations as well as (2.10) and (2.11) we can deduce (2.16) and (2.17) in a similar way. ∎

From (2.3) it is easy to see that if all m_{kn}, $1 \leqslant k \leqslant n$, $n \in \mathbb{N}$, are even then

$$\begin{aligned} &\lambda_{0k}(d\alpha) \leqslant \int_{x_{k+1}(d\alpha)}^{x_{k-1}(d\alpha)} d\alpha(x) \\ &(x_{-1}(d\alpha) := 1,\ x_{n+2}(d\alpha) := -1), \qquad n_0 \leqslant k \leqslant n_1. \end{aligned} \tag{2.19}$$

As an immediate consequence of Lemma 2 and the relation (2.19) we state

Corollary 1. *Assume that all* m_{kn}, $1 \leqslant k \leqslant n$, $n \in \mathbb{N}$, *are even. We have the inequality*

$$\begin{aligned} |\lambda_{jkn}(d\alpha)| \leqslant c d_{kn}^j \lambda_{0kn}(d\alpha) \leqslant c d_{kn}^j \int_{x_{k+1,n}(d\alpha)}^{x_{k-1,n}(d\alpha)} d\alpha(x), \\ 0 \leqslant j \leqslant m_{kn}^*, \quad n_0 \leqslant k \leqslant n_1. \end{aligned} \tag{2.20}$$

LEMMA 3. *Assume that all* m_{kn}, $1 \leqslant k \leqslant n$, $n \in \mathbb{N}$, *are even. For an arbitrary measure* $d\alpha$ *the relation*

$$\lim_{n\to\infty} \sum_{k=n_0}^{n_1} \sum_{j=0}^{m_{kn}^*} \lambda_{jkn}(d\alpha)\, f^{(j)}(x_{kn}(d\alpha)) = \int_{-1}^{1} f(x)\, d\alpha(x) \tag{2.21}$$

holds for all $f \in C^m[-1,1]$, *where* $m = \max_{n_0 \leqslant k \leqslant n_1, n \in \mathbb{N}} m_{kn}^*$.

Proof. Since (1.5) is exact for every polynomial $f \in \mathbf{P}_{N_n}$, by the well known Banach theorem it suffices to show

$$\sum_{k=n_0}^{n_1} \sum_{j=0}^{m_k^*} |\lambda_{jk}| \leqslant c < \infty. \tag{2.22}$$

This is indeed the case, because by (2.20) and (1.5)

$$\sum_{k=n_0}^{n_1} \sum_{j=0}^{m_k^*} |\lambda_{jk}| \leqslant c \sum_{k=n_0}^{n_1} \sum_{j=0}^{m_k^*} d_k^j \lambda_{0k} \leqslant 2^m(m+1) \sum_{k=n_0}^{n_1} \lambda_{0k}$$
$$= c2^m(m+1) \int_{-1}^{1} d\alpha(x). \quad \blacksquare$$

LEMMA 4. *Assume that all* m_{kn}, $1 \leqslant k \leqslant n$, $n \in \mathbb{N}$, *are even. Let* $d\alpha$ *be a measure on* $[-1,1]$. *If* $\int_a^b d\alpha(x) > 0$ ($[a,b] \subset [-1,1]$), *then for sufficiently large n the interval* $[a,b]$ *contains at least one zero* $x_{kn}(d\alpha)$.

Proof. Suppose to the contrary that there would exist a subsequence $\{n_i\}_{i=2}^{\infty}$, $n_i \to \infty$, such that the interval $[a,b]$ contains no zero $x_{k,n_i}(d\alpha)$. Choose $f \in C^m[-1,1]$ so that

$$f(x) \begin{cases} > 0, & x \in (a,b), \\ = 0, & x \notin (a,b). \end{cases}$$

Denote by n_{i0} and n_{i1} the corresponding numbers n_0 and n_1 for $n = n_i$, respectively. Then by Lemma 3

$$0 < \int_{-1}^{1} f(x)\, d\alpha(x) = \lim_{i\to\infty} \sum_{k=n_{i0}}^{n_{i1}} \sum_{j=0}^{m_{k,n_i}^*} \lambda_{j,k,n_i} f^{(j)}(x_{k,n_i}) = 0,$$

a contradiction. $\blacksquare$

Remark. This result extends Theorem 6.1.1 in [7, p. 107] concerning orthogonal polynomials.

The following result is an analogue for orthogonal polynomials [2, pp. 63–64].

Lemma 5. *Assume that all* m_{kn}, $1 \leqslant k \leqslant n$, $n \in \mathbb{N}$, *are even. Then the relation*

$$\lim_{n \to \infty} \max_{n_0 \leqslant k \leqslant n_1} \lambda_{0kn}(d\alpha) = 0 \tag{2.23}$$

holds if and only if

$$\alpha \in C[-1, 1]. \tag{2.24}$$

Proof. Assume that (2.23) is true. Let $y \in (-1, 1)$, say, $y \in (x_{k+1, n}, x_{k-1, n})$, $1 \leqslant k \leqslant n$. Then by (2.3)

$$\int_{x_{k+1, n}}^{x_{k-1, n}} d\alpha(x) \leqslant \lambda_{0, k+1, n} + \lambda_{0, k, n} + \lambda_{0, k-1, n}, \qquad \lambda_{0, n_0 - 1, n} := \lambda_{0, n_1 + 1, n} := 0.$$

Thus

$$\begin{aligned} \alpha(y+0) - \alpha(y-0) &\leqslant \alpha(x_{k-1, n}) - \alpha(x_{k+1, n}) \\ &\leqslant \lambda_{0, k+1, n} + \lambda_{0, k, n} + \lambda_{0, k-1, n} \to 0, \qquad n \to \infty. \end{aligned}$$

This proves continuity of $\alpha(x)$ at $x = y$. Similarly we can prove continuity of $\alpha(x)$ at $x = -1$ and $x = 1$.

Conversely, suppose that (2.24) holds. Let $\lambda_{0in} = \max_{n_0 \leqslant k \leqslant n_1} \lambda_{0kn}$. We may assume, passing to a subsequence if necessary, that as $n \to \infty$

$$\lambda_{0in} \to \lambda, \qquad x_{vn} \to y_v, \qquad v = i-1, i, i+1.$$

Then by (2.19)

$$\lambda \leqslant \int_{y_{i+1}}^{y_{i-1}} d\alpha(x).$$

It suffices to show $\int_{y_{i+1}}^{y_{i-1}} d\alpha(x) = 0$. Suppose not and let $\int_{y_{i+1}}^{y_{i-1}} d\alpha(x) > 0$. Then either $\int_{y_{i+1}}^{y_i} d\alpha(x) > 0$ or $\int_{y_i}^{y_{i-1}} d\alpha(x) > 0$ would occur. Assume without loss of generality that the first inequality occurs. Then $\int_{y_{i+1}+\varepsilon}^{y_i - \varepsilon} d\alpha(x) > 0$ holds for some $\varepsilon > 0$. Meanwhile by definition each interval $(x_{i+1, n}, x_{in})$ contains no zero x_{kn}. So for n large enough the interval $[y_{i+1} + \varepsilon, y_i - \varepsilon]$ contains no zero x_{kn}, contradicting Lemma 4. ▌

Lemma 6. *Assume that all* m_{kn}, $1 \leqslant k \leqslant n$, $n \in \mathbb{N}$, *are even. Let* $d\alpha$ *be a measure on* $[-1, 1]$ *such that* $\alpha \in C[-1, 1]$. *Then*

$$\lim_{n \to \infty} \sum_{k=n_0}^{n_1} \sum_{j=1}^{m_{kn}^*} |\lambda_{jkn}(d\alpha)| = 0. \tag{2.25}$$

Proof. By virtue of (2.20) and (2.23)

$$\sum_{k=n_0}^{n_1} \sum_{j=1}^{m_k^*} |\lambda_{jk}| \leqslant c \sum_{k=n_0}^{n_1} \sum_{j=1}^{m_k^*} d_k^j \lambda_{0k} \leqslant c[\max_{n_0 \leqslant k \leqslant n_1} \lambda_{0k}] \sum_{k=n_0}^{n_1} \sum_{j=1}^{m} d_k^j$$
$$\leqslant c2^{m-1}[\max_{n_0 \leqslant k \leqslant n_1} \lambda_{0k}] \sum_{j=1}^{m} \sum_{k=n_0}^{n_1} d_k$$
$$\leqslant cm2^{m+1}[\max_{n_0 \leqslant k \leqslant n_1} \lambda_{0k}] \to 0,$$

as $n \to \infty$. ▮

LEMMA 7. *Let* $x_{kn} = \cos\theta_{kn}$, $k = 0, 1, ..., n+1$, $n \in \mathbb{N}$, *be given in* (1.1). *If* (1.13) *is valid then*

$$|x_{kn} - x_{k+1,n}| \leqslant c\, \Delta_n(x_{in}), \qquad i = k, k+1, \quad 0 \leqslant k \leqslant n, \quad n \in \mathbb{N} \tag{2.26}$$

and

$$d_{kn} \leqslant c\, \Delta_n(x_{kn}), \qquad 0 \leqslant k \leqslant n+1, \quad n \in \mathbb{N}. \tag{2.27}$$

Proof. Let $i = k, k+1$. According to the mean value theorem for the derivatives by (1.13) we have that for some $\theta^* \in (\theta_k, \theta_{k+1})$

$$|x_k - x_{k+1}| = |\cos\theta_k - \cos\theta_{k+1}| = |(\theta_{k+1} - \theta_k)\sin\theta^*|$$
$$= |(\theta_{k+1} - \theta_k)\sin(\theta_i + \theta^* - \theta_i)|$$
$$= |(\theta_{k+1} - \theta_k)[\sin\theta_i \cos(\theta^* - \theta_i) + \cos\theta_i \sin(\theta^* - \theta_i)]|$$
$$\leqslant (\theta_{k+1} - \theta_k)[\sin\theta_i + |\sin(\theta^* - \theta_i)|] \leqslant \frac{c}{n}\left(\sin\theta_i + \frac{1}{n}\right).$$

Hence (2.26) follows. Inequality (2.27) directly follows from (2.26). ▮

3. PROOFS OF THEOREMS

3.1. *Proof of Theorem* 1. By (2.21) and (2.25)

$$\lim_{n\to\infty} Q_n(d\alpha; f) = \lim_{n\to\infty} \left[\sum_{k=n_0}^{n_1} \sum_{j=0}^{m_k^*} \lambda_{jk} f^{(j)}(x_k) - \sum_{k=n_0}^{n_1} \sum_{j=r_k+1}^{m_k^*} \lambda_{jk} f^{(j)}(x_k) \right]$$
$$= \int_{-1}^{1} f(x)\, d\alpha(x).$$

Equations (1.9) and (1.10) are direct consequences of (1.8). ▮

3.2. *Proof of Theorem* 2. Let $P_n \in \mathbf{P}_n$ satisfy (2.1) and (2.2) with $p = r$. Clearly

$$\int_{-1}^{1} P_n(x)\, d\alpha(x) = \sum_{k=n_0}^{n_1} \sum_{j=0}^{m_k^*} \lambda_{jk} P_n^{(j)}(x_k)$$
$$= Q_n(d\alpha; P_n) + \sum_{k=n_0}^{n_1} \sum_{j=r+1}^{m_k^*} \lambda_{jk} P_n^{(j)}(x_k).$$

Hence

$$\left| Q_n(d\alpha; f) - \int_{-1}^{1} f(x)\, d\alpha(x) \right|$$
$$= \left| \int_{-1}^{1} [P_n(x) - f(x)]\, d\alpha(x) + Q_n(d\alpha; f - P_n) - \sum_{k=n_0}^{n_1} \sum_{j=r+1}^{m_k^*} \lambda_{jk} P_n^{(j)}(x_k) \right|$$
$$\leqslant \left| \int_{-1}^{1} [f(x) - P_n(x)]\, d\alpha(x) \right| + |Q_n(d\alpha; f - P_n)|$$
$$+ \left| \sum_{k=n_0}^{n_1} \sum_{j=r+1}^{m_k^*} \lambda_{jk} P_n^{(j)}(x_k) \right|$$
$$:= S_1 + S_2 + S_3.$$

By (2.1)

$$S_1 \leqslant c \int_{-1}^{1} \Delta_n(x)^r\, \omega(f^{(r)}; \Delta_n(x))\, d\alpha(x) \leqslant c n^{-r} \omega(f^{(r)}; 1/n).$$

Applying (2.1) and (2.20)

$$S_2 \leqslant \sum_{k=n_0}^{n_1} \sum_{j \leqslant r} |\lambda_{jk}| \cdot |f^{(j)}(x_k) - P_n^{(j)}(x_k)|$$
$$\leqslant c n^{-r} \omega(f^{(r)}; 1/n) \sum_{k=n_0}^{n_1} \lambda_{0k} \sum_{j \leqslant r} d_k^j\, \Delta_n(x_k)^{-j}.$$

By means of (2.2) and (2.20)

$$S_3 \leqslant c n^{-r} \omega(f^{(r)}; 1/n) \sum_{k=n_0}^{n_1} \lambda_{0k} \sum_{j=r+1}^{m_k^*} d_k^j\, \Delta_n(x_k)^{-j}.$$

Substituting S_1, S_2, and S_3 into (3.1), we get (1.12). ∎

3.3. *Proof of Theorem* 3. Recalling

$$\sum_{k=n_0}^{n_1} \lambda_{0k} = \int_{-1}^{1} d\alpha(x),$$

(1.4) follows from (1.12) and (2.27). ∎

REFERENCES

1. B. D. Bojanov, D. Braess, and N. Dyn, Generalized Gaussian quadrature formulas, *J. Approx. Theory* **48** (1986), 335–353.
2. G. Freud, "Orthogonal Polynomials," Akad. Kiadó, Budapest; Pergamon, Oxford, 1971.
3. P. O. Runck, Bemerkungen zu den approximationssätzen von Jackson und Jackson–Timan, *in* "ISNM 10, Abstract Spaces and Approximation, Proc. of Conf. Oberwolfach, 1968" (P. L. Butzer and Sz. Nagy, Eds.), pp. 303–308, Birkhäuser, Basel, 1969.
4. Y. G. Shi, The generalized Markov–Stieltjes inequality for Birkhoff quadrature formulas, *J. Approx. Theory* **86** (1996), 229–239.
5. Y. G. Shi, On Hermite interpolation, *J. Approx. Theory*, in press.
6. Y. G. Shi, Generalized Gaussian quadrature formulas for Tchebycheff systems, *Far East J. Appl. Math.* **3** (1999), 153–170.
7. G. Szegő, "Orthogonal Polynomials," Amer. Math. Soc. Colloq. Publ., Vol. 23, Amer. Math. Soc., Providence, 1939.